Similarity between images

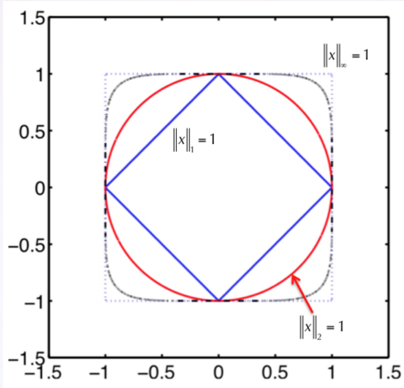
Need to define matrix norm $\|\cdot\|$ such that: for $\forall\, f, g \in \mathcal{I}$, we can define similarity between f and g as $\|f-g\|$.

Definition: A vector / matrix norm is a function $\|\cdot\| : \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) $\to \mathbb{R}$ so that for any $\vec{x}, \vec{y} \in \mathbb{R}^m$ (or $\mathbb{R}^{m\times n}$) and $\alpha \in \mathbb{R}$, we have:

1. $\|\vec{x}\| \geq 0$, $\|\vec{x}\| = 0$ iff $\vec{x} = 0$.
2. $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality)
3. $\|\alpha\vec{x}\| = |\alpha|\,\|\vec{x}\|$

Example:

$$\left.\begin{aligned}
&\bullet\ \|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i| \qquad \vec{x} = (x_1, x_2, \ldots, x_m)^T \\
&\bullet\ \|\vec{x}\|_2 = \left(\sum_{i=1}^{m} x_i^2\right)^{1/2} \\
&\bullet\ \|\vec{x}\|_\infty = \max_{i=1,2,\ldots,m} |x_i|
\end{aligned}\right\}\ \text{Vector norm}$$



Remark: In image processing, vector norm can be considered as matrix norm.

Suppose $A = \begin{pmatrix} a_{11} & a_{12} & & a_{1N} \\ a_{21} & a_{22} & & a_{2N} \\ \vdots & \vdots & \cdots & \vdots \\ a_{N1} & a_{N2} & & a_{NN} \end{pmatrix} \in M_{N\times N}(\mathbb{R}) \longrightarrow \vec{A} = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{N1} \\ \vdots \\ a_{1N} \\ a_{2N} \\ \vdots \\ a_{NN} \end{pmatrix}$

1st col of A

N^{th} col of A

Then: $\|A\|_1 = \|\vec{A}\|_1 = \sum_{i=1}^{N} \sum_{j=1}^{N} |a_{ij}|$

Given two images A and B, similarity between them can be measured by: $\sum_{i=1}^{N}\sum_{j=1}^{N} |a_{ij} - b_{ij}|$

<u>Another commonly used matrix norm</u>

<u>Definition: (Frobenius norm)</u> $\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}^2}$

Let $\vec{a}_j$ = j-th col of A. We have: $\|A\|_F = \sqrt{\sum_{j=1}^{n} \|a_j\|_2^2} = \sqrt{tr(A^T A)} = \sqrt{tr(AA^T)}$

where $tr(\cdot)$ = trace of the matrix.

Importance of defining correct norm

Figure 1: The images on the left and on the right are equally similar to the image in the middle in terms of the entrywise 1-norm. On the other hand, the image on the right is significant less similar to the image in the middle in terms of the entrywise 2-norm than the image on the left.

Figure 2: The images on the left and on the right are equally similar to the image in the middle in terms of the entrywise 2-norm. On the other hand, the image on the right is significant less similar to the image in the middle in terms of the entrywise 1-norm than the image on the left.

Representation of $\mathcal{O}$ by a matrix H:

Let $g = \mathcal{O}(f) \in M_{N\times N}(\mathbb{R})$.

Then: $g(\alpha,\beta) = \sum_{x=1}^{N} \sum_{y=1}^{N} h^{\alpha,\beta}(x,y)\, f(x,y)$ for $1 \le \alpha, \beta \le N$

So,

$$(*) \begin{cases} g(1,1) = h^{1,1}(1,1) f(1,1) + \dots + h^{1,1} f(1,N) + \dots + h^{1,1}(N,1) f(N,1) + \dots + h^{1,1}(N,N) f(N,N) \\ g(2,1) = h^{2,1}(1,1) f(1,1) + \dots + h^{2,1} f(1,N) + \dots + h^{2,1}(N,1) f(N,1) + \dots + h^{2,1}(N,N) f(N,N) \\ \vdots \\ g(\alpha,\beta) = h^{\alpha,\beta}(1,1) f(1,1) + \dots + h^{\alpha,\beta} f(1,N) + \dots + h^{\alpha,\beta}(N,1) f(N,1) + \dots + h^{\alpha,\beta}(N,N) f(N,N) \\ \vdots \\ g(N,N) = h^{N,N}(1,1) f(1,1) + \dots + h^{N,N} f(1,N) + \dots + h^{N,N}(N,1) f(N,1) + \dots + h^{N,N}(N,N) f(N,N) \end{cases}$$

N^2 equations, N^2 variables. LHS $= \begin{pmatrix} g(1,1) \\ g(2,1) \\ \vdots \\ g(N,1) \\ \vdots \\ g(1,N) \\ \vdots \\ g(N,N) \end{pmatrix} = \vec{g} \in \mathbb{R}^{N^2}$

Variables on RHS: $\begin{pmatrix} f(1,1) \\ f(2,1) \\ \vdots \\ f(N,1) \\ \vdots \\ f(1,N) \\ \vdots \\ f(N,N) \end{pmatrix} = \vec{f} \in \mathbb{R}^{N^2}$

So, (*) can be written in matrix form:

$$\vec{g} = H \vec{f} \qquad \left(H \in M_{N^2 \times N^2}(\mathbb{R}) \right)$$

H is called the <u>transformation matrix</u> representing $\mathcal{O}$.

Example 1.1 A linear operator is such that it replaces the value of each pixel by the average of its four nearest neighbours. Assume the image is repeated in all directions. Apply this operator $\mathcal{O}$ to a 3×3 image. Find the transformation matrix corresponding to $\mathcal{O}$.

Solution:

$$3 \times 3 \text{ image} = \begin{array}{ccccccccc|l}
 & & & f_{11} & f_{12} & f_{13} & & & & \leftarrow \text{Row } -3 \\
 & & & f_{21} & f_{22} & f_{23} & & & & \leftarrow \text{Row } -2 \\
 & & & f_{31} & f_{32} & f_{33} & & & & \leftarrow \text{Row } -1 \\
f_{11} & f_{12} & f_{13} & f_{11} & f_{12} & f_{13} & f_{11} & f_{12} & f_{13} & \leftarrow \text{Row } 1 \\
f_{21} & f_{22} & f_{23} & f_{21} & f_{22} & f_{23} & f_{21} & f_{22} & f_{23} & \leftarrow \text{Row } 2 \\
f_{31} & f_{32} & f_{33} & f_{31} & f_{32} & f_{33} & f_{31} & f_{32} & f_{33} & \leftarrow \text{Row } 3 \\
\uparrow & \uparrow & \uparrow & f_{11} & f_{12} & f_{13} & \uparrow & \uparrow & \uparrow & \leftarrow \text{Row } 4 \\
 & & & f_{21} & f_{22} & f_{23} & & & & \leftarrow \text{Row } 5 \\
\text{Col } -3 & \text{Col } -2 & \text{Col } -1 & f_{31} & f_{32} & f_{33} & \text{Col } 4 & \text{Col } 5 & \text{Col } 6 & \leftarrow \text{Row } 6
\end{array}$$

$$g_{22} = \frac{f_{12} + f_{21} + f_{23} + f_{32}}{4} \quad ; \quad g_{33} = \frac{f_{23} + f_{32} + f_{31} + f_{13}}{4}$$

etc ...

Write

$$\vec{g} = \begin{pmatrix} g_{11} \\ g_{21} \\ g_{31} \\ g_{12} \\ g_{22} \\ g_{32} \\ g_{13} \\ g_{23} \\ g_{33} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & 0 \\ & & & & \vdots & & & & \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{21} \\ f_{31} \\ f_{12} \\ f_{22} \\ f_{32} \\ f_{13} \\ f_{23} \\ f_{33} \end{pmatrix}$$

$h^{1,1}(3,1)$

$h^{2,1}(2,3)$

etc

$$g_{11} = \frac{f_{13} + f_{31} + f_{12} + f_{21}}{4}$$

$$g_{21} = \frac{f_{11} + f_{22} + f_{31} + f_{23}}{4}$$

By careful examination, we see that

$$\begin{bmatrix} 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\ 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 0 & 1/4 & 0 \\ 1/4 & 1/4 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 & 0 \\ 0 & 0 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 0 & 1/4 & 1/4 \\ 0 & 1/4 & 0 & 0 & 1/4 & 0 & 1/4 & 0 & 1/4 \\ 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 & 0 \end{bmatrix}$$

All entries are given by the point spread function $h^{\alpha,\beta}(x, y)$

__Example__ Consider $\theta : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ defined by:

$$\theta(f) = \underbrace{\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}}_{A} f \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{B} \quad \text{for all } f \in M_{N\times N}(\mathbb{R}).$$

Let $g = \theta(f)$. Then:

$$g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} f_{11} + 3f_{12} & 2f_{11} + 4f_{12} \\ f_{11} + 2f_{21} + 3f_{12} + 6f_{22} & 2f_{11} + 4f_{21} + 4f_{12} + 8f_{22} \end{pmatrix}$$

$$\begin{pmatrix} g_{11} \\ g_{21} \\ g_{12} \\ g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 3 & 6 \\ 2 & 0 & 4 & 0 \\ 2 & 4 & 4 & 8 \end{pmatrix} \begin{pmatrix} f_{11} \\ f_{21} \\ f_{12} \\ f_{22} \end{pmatrix}$$

$$\begin{pmatrix} 1\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} & 3\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \\ 2\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} & 4\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1A & 3A \\ 2A & 4A \end{pmatrix}$$

Remark: • Separable image transformation has a special structure.

• Let A and B be two matrices.
Kronecker product of A and B $= A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1N}B \\ a_{21}B & \cdots & a_{2N}B \\ \vdots & \vdots & \\ a_{N1}B & \cdots & a_{NN}B \end{pmatrix}$

$A = (a_{ij})_{1 \le i,j \le N}$

• In general, if $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB \quad \text{for all } f \in M_{N\times N}(\mathbb{R}), \text{ where } A, B \in M_{N\times N}(\mathbb{R})$$

Then, the transformation matrix of $\mathcal{O}$ is:

$$H = B^T \otimes A$$

• So, instead of storing $N^2 \times N^2 = N^4$ entries, we only need to store entries of A and B, which is $2N^2$ (much less storage)